OrientationJ: Theoretical Background

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Quantitative orientation analysis

The aim is to characterize the orientation and isotropy properties of a local area of interest (Region of Interest, ROI) in an image. To that end, we first define the weighted inner product

$$\langle f, g \rangle_w = \iint_{\mathbb{R}^2} w(x, y)\, f(x, y)\, g(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{1}$$

where $w(x, y) \geq 0$ is a weighting function that specifies the area of interest. It is typically a normalized square window of size L centered on a location of interest (x_0, y_0), or a Gaussian window of standard deviation σ_w. The norm associated with this inner product is $\|f\|_w = \sqrt{\langle f, f \rangle_w}$. Next, we consider the derivative in the direction specified by the unit vector $\mathbf{u}_\theta = (\cos\theta, \sin\theta)^\top$, which is given by

$$D_{\mathbf{u}_\theta} f(x, y) = \mathbf{u}_\theta^\top \, \nabla f(x, y), \tag{2}$$

where $\nabla f = (f_x, f_y)^\top$ is the gradient of the image under consideration. We are now interested in finding the direction $\mathbf{u}$ along which the directional derivative is maximized over the ROI. It is given by

$$\mathbf{u}_{\max} = \arg \max_{\|\mathbf{u}\|=1} \; \|D_{\mathbf{u}} f\|_w^2. \tag{3}$$

A standard inner-product manipulation then yields

$$\|D_{\mathbf{u}} f\|_w^2 = \langle \mathbf{u}^\top \nabla f, \nabla f^\top \mathbf{u} \rangle_w = \mathbf{u}^\top \mathbf{J}\, \mathbf{u}, \qquad \mathbf{J} = \langle \nabla f, \nabla f^\top \rangle_w = \begin{pmatrix} \langle f_x, f_x \rangle_w & \langle f_x, f_y \rangle_w \\ \langle f_x, f_y \rangle_w & \langle f_y, f_y \rangle_w \end{pmatrix}, \tag{4}$$

where $\mathbf{J}$ is the so-called *structure tensor*, a 2×2 symmetric positive-semidefinite matrix. The solution of the optimization problem (3) is obtained by setting the derivative of $\mathbf{u}^\top \mathbf{J}\, \mathbf{u} + \lambda(1 - \mathbf{u}^\top \mathbf{u})$ with respect to $\mathbf{u}$ to zero, which yields the eigenvector equation

$$\mathbf{J}\, \mathbf{u} = \lambda\, \mathbf{u}. \tag{5}$$

This implies that the first eigenvector $\mathbf{e}_1$ of $\mathbf{J}$ gives the direction of maximal gradient energy; the corresponding eigenvalue is $\lambda_1 = \max \|D_{\mathbf{u}} f\|_w^2$. Conversely, the directional derivative is minimized along the second eigenvector $\mathbf{e}_2$, with $\lambda_2 = \min \|D_{\mathbf{u}} f\|_w^2$. Since the gradient is perpendicular to the local structures, the visible structures (fibres, edges) are aligned with $\mathbf{e}_2$; this is the orientation reported by OrientationJ. The structure tensor therefore contains all the relevant directional information of the ROI.

Features and tensor invariants

With the eigenvalues $\lambda_1 \geq \lambda_2 \geq 0$, the mean $\bar{\lambda} = \frac{1}{2}\operatorname{tr}(\mathbf{J})$, and the deviator $\mathbf{s} = \mathbf{J} - \frac{1}{2}\operatorname{tr}(\mathbf{J})\, \mathbf{I}$, OrientationJ computes the following features:

- **Orientation**: $\theta = \dfrac{1}{2}\arctan\left(\dfrac{2\langle f_x, f_y \rangle_w}{\langle f_y, f_y \rangle_w - \langle f_x, f_x \rangle_w}\right) \in [-\pi/2, \pi/2]$ (direction of $\mathbf{e}_2$, along the structures)

- **Energy**: $E = \mathrm{tr}(\mathbf{J}) = \lambda_1 + \lambda_2 \in [0, \infty)$
- **Coherency**: $C = \dfrac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} = \dfrac{\sqrt{\left(\langle f_y, f_y\rangle_w - \langle f_x, f_x\rangle_w\right)^2 + 4\,\langle f_x, f_y\rangle_w^2}}{\langle f_x, f_x\rangle_w + \langle f_y, f_y\rangle_w} \in [0, 1]$
- **Directionality**: $J_2 = \frac{1}{2}\,\mathrm{tr}(\mathbf{s}^2) = \frac{1}{4}(\lambda_1 - \lambda_2)^2 = \frac{1}{4}\,C^2E^2 \in [0, \infty)$
- **Fractional anisotropy**: $\mathrm{FA} = \dfrac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} = \dfrac{\sqrt{2}\,C}{\sqrt{1 + C^2}} \in [0, 1]$

The coherency indicates whether the local image features are oriented or not: C is 1 when the local structure has one dominant orientation, and C is 0 if the image is essentially isotropic in the local neighborhood. The fractional anisotropy [5] carries the same information through the one-to-one map $\mathrm{FA} = \sqrt{2}\,C/\sqrt{1 + C^2}$, but normalizes the eigenvalue contrast by the Frobenius norm of the tensor, following the usage established in diffusion-tensor imaging. The directionality J_2 is the second invariant of the deviator (the von Mises invariant [4]); it is an *unnormalized* measure that grows with both contrast and alignment.

Table 1 summarizes the features and invariants and the correspondence between them. Two independent scalars fix the tensor up to rotation; complete sets include (λ_1, λ_2), (I_1, J_2), (E, C), and (E, FA). Table 2 evaluates every feature on canonical eigenvalue pairs, from the ideal oriented case $(1, 0)$ to the isotropic case $(1, 1)$.

Implementation notes

In OrientationJ, the weighting function w is a Gaussian window whose standard deviation σ_w ("local window") is set in the dialog; the gradient is computed by cubic-spline interpolation by default. A small regularization ϵ is added to the denominators of C and FA to avoid division by zero in flat regions.

Coherency and fractional anisotropy are dimensionless and naturally bounded in $[0, 1]$; they are displayed as computed. Energy and directionality are unbounded (units g^2 and g^4); since version 2.1.0, they are either linearly rescaled to $[0, 1]$ for display (option *Scale [0..1]*, the default) or shown with their raw values (option *No scale*). The same option applies to the channels used to build the HSB/RGB color survey, where channel values are clamped to $[0, 1]$.

References

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Table 1: Local orientation features and tensor invariants in 2D. The gradient structure tensor is $\mathbf{J} = \langle \nabla I\, \nabla I^\top \rangle_w$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq 0$, mean $\bar{\lambda} = \frac{1}{2} I_1$, and $g = \|\nabla I\|$.

	Components	Eigenvalues	Correspondence	Name / Interpretation
Tensor				
Tensor $\mathbf{J}$ positive-semidefinite	$\begin{pmatrix} J_{xx} & J_{xy} \\ J_{xy} & J_{yy} \end{pmatrix}$	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	—	Gradient structure tensor [1]
Orientation θ radians, $[-\pi/2, \pi/2]$	$\frac{1}{2}\arctan\left(\frac{2J_{xy}}{J_{yy} - J_{xx}}\right)$	$\mathbf{e}_2$	—	Principal direction, nematic director
Features				
Energy E g^2, $[0, \infty)$	$J_{xx} + J_{yy}$	$\lambda_1 + \lambda_2$	$E = I_1$	Gradient energy [2]
Coherency C dimensionless, $[0, 1]$	$\frac{\sqrt{(J_{yy} - J_{xx})^2 + 4J_{xy}^2}}{J_{xx} + J_{yy}}$	$\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$	$C = \frac{2\sqrt{J_2}}{I_1}$	Alignment index, nematic order parameter; 0 isotropic, 1 fibre [2, 3]
Properties				
Deviator $\mathbf{s}$ $\mathrm{tr}(\mathbf{s}) = 0$	$\mathbf{J} - \frac{1}{2}\,\mathrm{tr}(\mathbf{J})\,\mathbf{I}$	$\begin{pmatrix} \lambda_1 - \bar{\lambda} & 0 \\ 0 & \lambda_2 - \bar{\lambda} \end{pmatrix}$	—	Deviatoric part of $\mathbf{J}$
Intensity I_1 g^2, $[0, \infty)$	$\mathrm{tr}(\mathbf{J}) = J_{xx} + J_{yy}$	$\lambda_1 + \lambda_2$	$I_1 = E$	First invariant
Directionality J_2 g^4, $[0, \infty)$	$\frac{1}{2}\,\mathrm{tr}(\mathbf{s}^2) = \frac{1}{4}(J_{xx} - J_{yy})^2 + J_{xy}^2$	$\frac{1}{4}(\lambda_1 - \lambda_2)^2$	$J_2 = \frac{1}{4} C^2 E^2$	Second deviatoric invariant [4]
Distortion energy σ g^2, $[0, \infty)$	$\sqrt{2}\,\|\mathbf{s}\|$	$\lambda_1 - \lambda_2$	$\sigma = 2\sqrt{J_2} = I_1\,\mathrm{RA}$	Equivalent uniaxial magnitude
Anisotropy indices				
Relative anisotropy RA dimensionless, $[0, 1]$	$\frac{\|\mathbf{s}\|}{\sqrt{2}\,\bar{\lambda}}$	$\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$	$\mathrm{RA} = C$	Coefficient of variation of the λ_i [5]
Fractional anisotropy FA dimensionless, $[0, 1]$	$\frac{\sqrt{2}\,\|\mathbf{s}\|}{\|\mathbf{J}\|}$	$\frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}$	$\mathrm{FA} = \frac{\sqrt{2}\,C}{\sqrt{1 + C^2}}$	Degree of anisotropy [5]

Table 2: Typical cases in 2D. Features of Table 1 evaluated from the eigenvalues, each by the expression given in its Eigenvalues column there; $\bar{\lambda} = \frac{1}{2} I_1$.

	Structure	$E = I_1$	J_2	σ	C	RA	FA
(1, 0)	ideal oriented	1.000	0.250	1.000	1.000	1.000	1.000
(5, 0.2)	strong	5.200	5.760	4.800	0.923	0.923	0.959
(3, 1)	oriented	4.000	1.000	2.000	0.500	0.500	0.632
(2, 1)	moderate	3.000	0.250	1.000	0.333	0.333	0.447
(1, 0.5)	weak	1.500	0.062	0.500	0.333	0.333	0.447
(1, 0.9)	near-iso.	1.900	0.002	0.100	0.053	0.053	0.074
(1, 1)	isotropic	2.000	0.000	0.000	0.000	0.000	0.000
(0, 0)	flat	0.000	0.000	—	—	—	—